Relativistic kinematics of the wave packet

Hans de Vries*

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1 Relativistic kinematics of the wave packet

We start with the basic wave equation (1) which is at the base of most of quantum mechanics, including Heisenberg's uncertainty principle, Schrdinger's equation, The relativistic Klein Gordon equation, et-cetera. The Energy is found to be dependent on the phase change rate in time given by the frequency f:

$$E = hf \tag{1}$$

The wave equation (2) for the momentum p follows automatically from the theory of Special Relativity which will be shown. The momentum p is found to dependent on the phase change rate over space resulting in the deBroglie matter wavelength λ:

$$p = h/\lambda \tag{2}$$

The deBroglie wave-length is caused purely by the relativistic effect of non - simultaneity: the time-shift we see if we look at the particle from a reference frame in which it is not at rest. The deBroglie wave-length is a relativistic effect even though it occurs at speeds of centimeters per second. So equation (2) is **not** a separate law but follows directly from (1).

2 The wave packet at rest

The wave function of a particle in its rest frame is represented by (3), where Q_x is a localized Quantum wave packet. E_0 is the energy belonging to its rest mass m_0. The particle viewed from its rest-frame has an equal (complex) phase over all of space: This means that a particle at rest has a deBroglie wavelength λ of ∞.

$$\text{particle at rest:} \quad \Psi \;=\; Q_x\, e^{-i2\pi f\mathbf{t}} \;=\; Q_x\, e^{-iE_0\mathbf{t}/\hbar} \tag{3}$$

The effect of the wave-packet function Q_x is that the particle is localized. The Fourier transform $\mathcal{F}\{Q_x\}$ of Q will introduce extra spacial frequencies around the central wavelength λ. For the next few sections we will work with pure frequencies. We will include Q again at the section which discusses the group speed of the deBroglie wave.

*hansdevries@chip-architect.com

3 The moving wave packet

The relativistic time shift seen from a reference frame other then the rest frame produces different phase shifts in $e^{iE0t/\hbar}$ at different x locations which then manifest them self as the deBroglie wave length, a complex phase changing over space with a wave length λ.

$$\text{moving particle:} \quad \Psi \ = \ e^{i2\pi \mathbf{x}/\lambda} \ e^{-i2\pi f \mathbf{t}} \ = \ e^{ip\mathbf{x}/\hbar \,-\, iE\mathbf{t}/\hbar} \tag{4}$$

We can simply derive the formula above from (3) if we substitute t with t' from the Lorentz transformation:

$$t' = \gamma \left(-\frac{vx}{c^2} + t \right) \tag{5}$$

$$e^{-E_0 \mathbf{t}/\hbar} \ = \ e^{-im_0c^2 \mathbf{t}/\hbar} \Rightarrow e^{-im_0c^2 \ \gamma \left(-vx/c^2 + t\right)/\hbar} \tag{6}$$

$$= \ e^{i\gamma m_0 vx/\hbar \,-\, i\gamma m_0 c^2 t/\hbar} \ = \ e^{ipx/\hbar \,-\, iEt/\hbar} \tag{7}$$

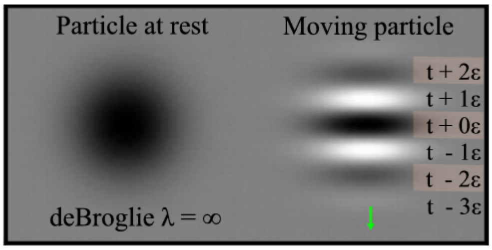

Figure 1: The deBroglie wave as a result of non-simultaneity

With the relativistic momentum $p = \gamma m_0 v$ and the relativistic energy $E = \gamma m_0 c^2$ we get our expression (4) for the moving particle. We have derived the wave behavior of momentum from the wave behavior of energy. The image shows a particle at rest with $\lambda = \infty$ (localized by the function Q) and a particle moving downwards with an indication of the time bands in the rest frame of the particle

4 The $> c$ phase speed of the deBroglie wave

The deBroglie wave length is inversely proportional to the speed and becomes infinite in the rest frame. This simply means that the phase is equal everywhere in the restframe. The speed with which the phase "moves" $f\lambda$ in the rest frame thus becomes infinite as well. The phase speed is the inverse of the material speed v. The phase speed only equals the material speed in the limit of c:

$$\text{deBroglie phase speed:} \quad v_\psi \ = \ f\lambda \ = \ \frac{E}{p} \ = \ \frac{c^2}{v} \tag{8}$$

This result, although correct and logical after the derivation of $p = h/\lambda$ from $E = hf$ contradicts intuition. What is behind this is that we can not interprete the deBroglie wave as a phenomena which propagates with a certain speed through a media which is either at rest or has it's own specific speed. This does not work. In the next section however we'll show how by a simple re-arrangement of equation (4) we get all the physical behavior we want.

5 The group speed of the deBroglie wave

We have shown that one doesn't get logical physical behavior if one tries to interpret a Quantum Mechanical wave as something which propagates trough a medium . Well, this is hardly surprising from the viewpoint of Special Relativity.

We do get the expected physical behavior however if we assume that the wave function itself moves with the physical speed v of the wave packet. That is, as far as one can speak in terms of 'media', one must assume that any 'medium' moves along with the wave packet at the same speed. One can **not** say that the wave packet propagates through the 'media' Now for the math we start with:

$$\text{moving wave packet:} \quad Q_{x,t} \;\; e^{ipx/\hbar} \;\; e^{-iEt/\hbar} \tag{9}$$

Which we have split into three parts. First we want to express the localized packet Q more explicitly as a something which moves with a speed v and hence is Lorentz contracted by a corresponding gamma. We do so by defining:

$$Q_{x,t} \;\; = \;\; Q(\gamma(x - vt)) \tag{10}$$

We now want to do the same for the second part of equation (9) which describes the phase change over space. We want to make it move with a physical speed v so we can view Q and the second term as a single combination which moves along with speed v. We already have the gamma factor included since:

$$e^{ipx/\hbar} \;\; = \;\; e^{i\gamma m_0 vx/\hbar} \tag{11}$$

(See equation (7), To make it physically moving at speed v we need to lend some from the third term to obtain the $-vt$ part of the factor $(x - vt)$. To do so we split the exponent of the third term as follows:

$$-iE\ t/\hbar \; = \; -\ iE\ t/\hbar \left(\frac{v^2}{c^2}\right) \; - \; iE\ t/\hbar \left(1 - \frac{v^2}{c^2}\right) \tag{12}$$

The first half we move to the space phase phase so we get:

$$e^{ipx/\hbar} \;\; e^{-iEt/\hbar} \;\; \Rightarrow \;\; e^{im_0 v\gamma(x - vt)/\hbar} \;\; e^{-\ im_0 c^2 t/(\hbar\gamma)} \tag{13}$$

With this we can write the re-arranged expression (9) for the moving wave packet:

$$Q(\gamma(x - vt)) \;\; e^{ip_0\gamma(x - vt)/\hbar} \;\; e^{-\ iE_0 t/(\hbar\gamma)} \tag{14}$$

$$= \;\; W(\gamma(x - vt)) \;\; e^{-\ iE_0 t/(\hbar\gamma)} \tag{15}$$

Where W is the combined Lorentz contracted function moving with speed v. The phase variation with time represented by the last factor is now to be understood as taken over the actual trajectory over the wave packet. It correctly corresponds with the time dilation, which is predicted by special relativity to be a factor γ slower as for the particle at rest.

Both the Lorentz contraction (with factor gamma) and the phase variation with x are the result of the non-simultaneity of Special Relativity. To see this we can imagine that we instantaneously "freeze" a bypassing traveler. Walking around him we can now see him "hanging in the air", indeed being contracted in the direction in he was moving.

The traveler however will complain that his front was stopped first, before his back was frozen, and argues that this is the reason of his compressed state. The same is true for the phase. The phase of the traveler does not vary with x in his rest frame. However since (seen from his rest frame) we froze his front first and his back later. We end up with the phase variation over x given by the second part of equation (14)

The remark made that we can not interpret the deBroglie wave as propagating through a media and, as far as one can speak in terms of 'media', one must assume that any 'medium' moves along with the wave packet at the same speed. This condition is also a requirement for the stability of the wave packet.

By multiplying with Q we increase the spectrum around the central wavelength to a non-zero width via its Fourier transform $\mathcal{F}\{Q_x\}$. The stability of the wave packet requires that all frequencies move at the same speed. If this is not the case then the wave packet would disperse in the direction of motion but remain constant in the orthogonal directions. Such an asymmetric behavior dependent on ones choice for the speed is in conflict with special relativity.

If one can speak of any propagation through a 'media' at all then it only serves to make the wave function coherent over space. It would then provide a mechanism to make the phase equal over space as seen from its rest frame.

6 The relativistic rotation of the wave front

A remarkable amount of physics can be extracted from the simple rule that the wave front is always at right angles with the physical velocity, regardless of the reference frame. This gives us another means of determining the material speed.

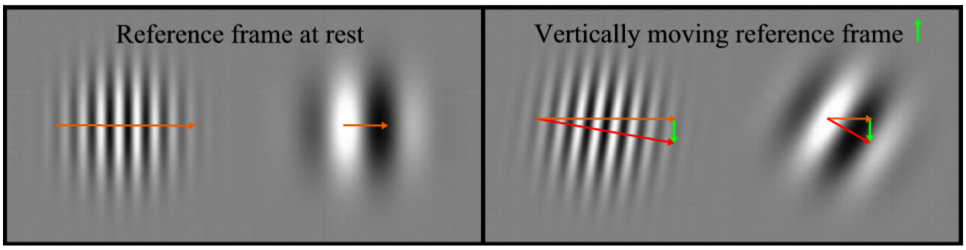

Figure 2: The Rotation of the Wavefronts

The left half above above shows a fast particle chasing a slower particle with equal mass. The fast particle has a shorter deBroglie wavelength. The phase speed of the faster particle is slower (as given by c^2/v) compared to the slower particle.

At the right half we see the same scene from a reference frame moving upwards. The extra motion has a larger influence on the slower moving particle. Its relative motion changes downwards more than the faster particle. As one can see, the combination of Special Relativity and Quantum Mechanics makes sure that the wavefronts are exactly at right angles with the physical speed, exactly as one would intuitively expect.

It is only Special Relativity which can rotate wavefronts, and it does so for both light and matter waves. A Galilean transformation keeps the wavefronts always directed in the same direction! The mechanism through which Special Relativity manages this is again via the non-simultaneity of time.

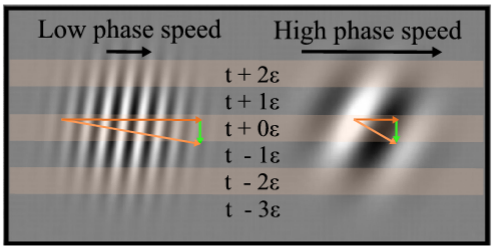

Figure 3: The time zones of Special Relativity

The time in the moving frame has progressed further in the upper time bands and less in the lower. Horizontally the phase has shifted further in the upper and less in the lower bands. The result is that the wavefront becomes skewed. The wavefront of the slower particle which has a higher phase speed (c^2/v) becomes more skewed and rotates further. Just as it should be to keep the wavefront at right angles with physical speed.

So it's Special Relativity which rotates the wavefront while it is the Quantum Mechanical deBroglie wave with its phase speed of c^2/v which rotates the wavefront of a slow particle more than that of a faster one. This mechanism works equally well for light waves which represent the limit where: group-speed = phase-velocity = c.